An Analysis of the Vertistat Gravity Gradient Satellite Orientation

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The Vertistat is a device that passively orients a satellite in a near circular orbit into a specific alignment with the local earth vertical. Basically, it consists of three long, orthogonally placed rods. The vertical rod, which is longest, causes a restoring torque due to the slight change in gravitational force along it. The three rods are interconnected by a spring-damper arrangement to provide damping of the satellite librations. This paper covers two important Vertistat performance areas, small deviation stability, and orientation error due to orbit eccentricity. Stability results include an analytical method for selection of Vertistat parameter values yielding optimum results for all altitudes. Analog computer time history data are given for these values; the data indicate that the device can remove an initial small deviation error within an orbit. It is shown that orbit eccentricity induces an error of 1.5°/0.01 of orbit eccentricity at any altitude.

Nomenclature

K_s	= spring constant, ft-lb/rad
K_D	= damping constant, ft-lb-sec/rad
K_s''', K_D'''	= spring and damper constants, β degree of freedom
K_{s_1}'', K_{D_1}''	= spring and damper constants, α degree of freedom
K_{s_2}'', K_{D_2}''	= spring and damper constants, γ degree of freedom
J_1, J_2	= axial and transverse moments of inertia of a long slender body, respectively, slug-ft^2
J_a^i, J_b^i	= axial and transverse moments of inertia, respectively, of Vertistat rods, slug-ft^2; "i" is a single, double, or triple prime corresponding to the primary (rod plus satellite), α and β rods, respectively
t	= time, sec
T	= gravity gradient torque, ft-lb
T_ϵ', T_ϵ''	= torque due to orbit eccentricity on the primary and secondary rods, respectively, ft-lb
α, β, γ	= angles defining attitude of the α and β rods relative to the satellite (Fig. 3), rad
$\Delta\xi_{ss}$	= maximum steady state sinusoidal variation of ξ due to orbital eccentricity, deg
δ, δ_1, δ_2	= damping ratio
ϵ	= orbital eccentricity
η, ξ, ϕ	= Euler angles defining satellite attitude (Fig. 2), rad
θ	= angle between J_1 axis and local vertical, rad
ν	= frequency of body oscillation, cps
τ	= $\omega_0 t$
τ_ϕ	= $\frac{4}{3}^{1/2}\tau$
ω_0	= orbital angular rate, rad/sec
ω, ω_1, ω_2	= undamped natural frequencies, rad/sec

Introduction

THE Vertistat employs the gravity gradient principle to passively orient a satellite in a near circular orbit in alignment with the local earth vertical.[1-3] Three long slender rods are interconnected by a spring-damper arrangement to provide damping of satellite librations. The vertical rod, or main boom, is longest; it provides the restoring torque due to the gravity gradient. This paper covers two important study areas, namely, small deviation stability and the influence of orbit eccentricity on alignment error.

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The equations of motion of the device are nonlinear differential equations. However, when these equations are restricted to small deviations off the local vertical, they reduce to linear differential equations. Straight-forward analysis of small disturbance stability is then possible in terms of the familiar indices of damping ratio and damping time. A time history derived from analog data is given for a special case corresponding to a near optimum selection of Vertistat parameters. This optimum selection, derived from analog computer parameter search, is then expressed analytically to yield parameter selection capability for all altitudes.

In a circular orbit, with no external perturbing torques acting on the Vertistat, the device will precisely align with local vertical. Orbit eccentricity is a perturbing effect that induces an error in alignment that is calculated herein.

Vertistat Functional Description

Assuming that the magnitude of the moment of inertia about one axis of the body, say J_1, is much smaller than the others, J_2 (e.g., a long slender rod), the gravity gradient torque, neglecting J_1 would be

$$T = (3\omega_0^2/2)J_2 \sin 2\theta \tag{1}$$

Thus a long slender rod tends to be aligned with the local vertical. Indeed it should act very much like a neutrally stable pendulum continuously oscillating at a frequency

$$\nu = (3)^{1/2}\omega_0/2\pi \text{ cps} \tag{2}$$

when θ is a small angle, so that $\sin 2\theta \simeq 2\theta$.

From Eq. (1) it is seen that the torque, as a function of alignment angle with the local vertical, increases from zero at alignment of the rod with local vertical to a maximum at $\theta =$ 45° and then reduces to zero again when the rod is perpendicular to local vertical. Figure 1 is a photograph of an implemented prototype.[4] Three long, slender, orthogonal rods are located symmetrically about a central support structure, which is rigidly mounted to the satellite. All rods are shown slightly extended. The vertical rod (only the top half of it is shown) is the longest; it is the primary boom, which delivers a restoring torque when not aligned with local vertical. One of the shorter rods, called the β boom, is swiveled to be free to rotate about an axis perpendicular to the primary boom axis. The other shorter rod, termed the α rod, is gimbaled to be free to rotate about an axis perpendicular to the primary rod axis and also about the primary rod axis. In each of the three rotation directions, damper and spring actions take place between the primary, α, and β rods.

When the primary rod axis is aligned with local vertical, the α and β rods are perpendicular to vertical, their position being maintained by the springs. Then according to Eq. (1) the α and β rods do not experience any gravity torque (because $\theta = 90°$). Any deviation of the main rod from local vertical causes a restoring gravity torque. If the α and β rods were not present, the main rod would oscillate at the frequency ν [Eq. (2)]. However, as will be shown below, the α and β rods together with their spring-damper suspensions cause this oscillation to damp.

Vertistat Equations of Motion

A complete set of equations of motion are derived in Ref. 1. These equations are lengthy and quite nonlinear. However, if we assume that there will be only small deviations about the local vertical, then the equations may be linearized. Essentially the derivations of the equations entail the use of Euler's equations of motion applied to each of the three separate bodies making up the Vertistat.

Figures 2 and 3 define the parameters and variables involved. The orbit frame of reference (axes 1, 2, 3) is defined relative to an inertial frame, with origin at the earth center in Fig. 2. The axis system, 1′, 2′, 3′, fixed to the primary rod, centered at the rod center of mass and with the 1′ axis along the primary rod axis, is defined relative to the orbital frame of reference by the Euler angle set η, ξ, ϕ. These quantities are the main variables of interest because they represent the errors in alignment in local vertical. Figure 3 defines the notation for suspension of the α and β rods. Two more coordinate frames are introduced, each fixed to one rod and defined relative to the primed coordinate reference (the primary rod). Note the α and γ variables, as defined, are the degrees of freedom of the α rod relative to the primary. Similarly, the β variable is the single degree of freedom of the β rod relative to the primary. In each case there are parallel combinations of spring (K_s) and damper (K_D) constants.

The standard form of the Euler equations of motion were applied with a moving axis system fixed to the three rods centered at the center of mass (the center of mass of all booms are common); these equations are nonlinear and lengthy but eventually reduce to six equations in six unknowns, ξ, η, ϕ, α, β, γ. For purposes of stability analysis these equations were linearized by assuming small deviations about the statically stable position where all six variables are zero.[5] This restriction places the primary rod in alignment with local vertical, the α and β rods perpendicular to the primary and to each other, with the α rod long axis within the orbit plane.

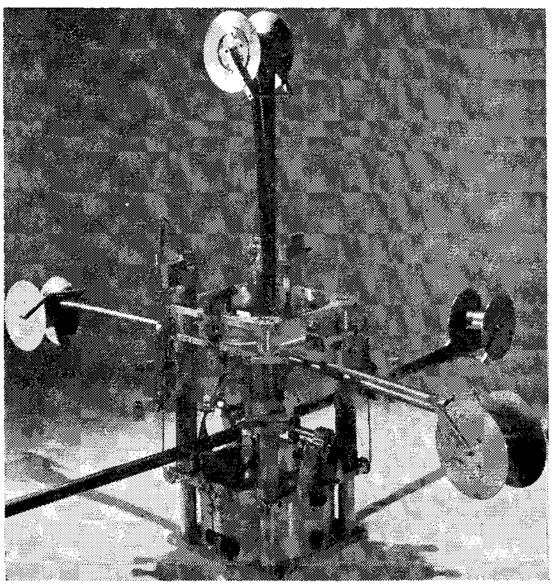

Fig. 1 Prototype model of Vertistat.

The small disturbance equations then are where

$$\Delta\ddot{\xi} + 3\left(1 - \frac{J_a'}{J_b'}\right)\Delta\xi + \frac{J_b''}{J_b'}\left(\frac{K_{s1}''}{J_b''\omega_0^2}\Delta\alpha + \frac{K_{D1}''}{J_b''\omega_0}\Delta\dot{\alpha}\right) = 2\epsilon\sin\tau \quad (3)$$

$$\Delta\ddot{\xi} - 3\Delta\xi - \Delta\ddot{\alpha} + \left(3 - \frac{K_{s1}''}{J_b''\omega_0^2}\right)\Delta\alpha - \frac{K_{D1}''}{J_b''\omega_0}\Delta\dot{\alpha} = 2\epsilon\sin\tau \quad (4)$$

$$\Delta\ddot{\phi} + 4\left(1 - \frac{J_a'}{J_b'}\right)\Delta\phi + \frac{J_a'}{J_b'}\Delta\dot{\eta} - \frac{J_b'''}{J_b'}\left(\frac{K_s'''}{J_b'''\omega_0^2}\Delta\beta + \frac{K_D'''}{J_b'''\omega_0}\Delta\dot{\beta}\right) = 0 \quad (5)$$

$$\Delta\ddot{\beta} - \left(4 - \frac{K_s'''}{J_b'\omega_0^2}\right)\Delta\beta + \frac{K_D'''}{J_b'''\omega_0}\Delta\dot{\beta} + \Delta\ddot{\phi} - 4\Delta\phi + 2\Delta\dot{\eta} = 0 \quad (6)$$

$$-\left(\frac{J_a'}{J_b'} + \frac{J_b''}{J_b'} + \frac{J_b'''}{J_b'}\right)\Delta\ddot{\eta} + \left(2\frac{J_b'''}{J_b'} + \frac{J_a'}{J_b'}\right)\Delta\dot{\phi} + \left(\frac{J_b'''}{J_b'} - \frac{J_b''}{J_b'}\right)\Delta\eta - \frac{J_b''}{J_b'}(\ddot{\gamma}\Delta + \Delta\gamma) + 2\frac{J_b'''}{J_b'}\Delta\dot{\beta} = 0 \quad (7)$$

$$\Delta\ddot{\eta} + \Delta\eta + \Delta\ddot{\gamma} + \left(1 + \frac{K_{s2}''}{J_b''\omega_0^2}\right)\Delta\gamma + \frac{K_{D2}''}{J_b''\omega_0}\Delta\dot{\gamma} = 0 \quad (8)$$

$\tau = \omega_0 t$, and all derivatives are with respect to τ. The time t is in seconds when ω_0 (the orbital rate) is given in radians per second; τ is then a unit of time specifically related to orbit period. When this time base change is introduced and the parameters are combined as shown in the preceding equations, a solution at one orbit altitude (corresponding to a particular value of ω_0) is applicable to all altitudes.

The right-hand sides of the equations represent a known sinusoidal function of time corresponding to a torque induced by orbit eccentricity to be explained later. Our first concern is dynamic stability, for which case this torque is set to zero. It is noted that the variables $\Delta\alpha$ and $\Delta\xi$ (pitch) appear only in Eqs. (3) and (4), whereas $\Delta\phi$ (roll), $\Delta\eta$ (yaw), $\Delta\gamma$, and $\Delta\beta$ appear only in Eqs. (5–8). Therefore, for small deviations, pitch motion analysis is separable from roll and yaw motions. If all terms in Eq. (3) except the first two are ignored, and the primary rod is long and slender ($J_a'/J_b' \rightarrow 0$), then the primary rod is neutrally stable in pitch (ξ) and oscillates at a frequency $(3)^{1/2}$ times the orbit frequency (ν). The α rod plus spring damper provides damping to the otherwise nondamped case.

It is emphasized that J_b' is the total moment of inertia of the satellite plus primary rod about a perpendicular to the rod long axis. The rod is long in comparison with satellite dimensions such that the satellite inertia contribution would be negligible in most cases. Conversely the moment of inertia of the satellite plus primary rod along the rod axis is almost entirely due to the satellite in a practical case due to the long slender shape of the rod. Thus, unless the satellite itself has some unusual shape, J_a' would correspond to the satellite moment of inertia and J_b' would correspond to the moment of inertia of the rod ignoring the satellite. Similarly in roll (ϕ) [Eq. (5)], ignoring all terms involving β and γ, it is noted that the primary rod will be neutrally stable in roll at twice orbital frequency. Further, from Eq. (7) it is noted that the primary rod will be stable in yaw only when $J_b'' > J_b'''$, which means that the static situation assumed as the initial orientation is only possible when the α rod is longer than the

β. The oscillation frequency depends upon the ratio of α to β rod inertias.

Note the organization of parameters where the spring and damping parameters are the only ones containing the orbit altitude parameter ω_0. This arrangement enables search for the damping $(K_D/J_b\omega_0)$ and spring $(K_s/J_b\omega_0^2)$ parameters such that once an optimum value is found the results are applicable for *any altitude.*

Stability

Pitch

The characteristic equation for pitch stability is obtained from Eqs. (3) and (4). This equation, expressed in terms of usual stability analysis parameters δ, the damping ratio, and ω/ω_0, the normalized undamped oscillation frequency is

$$\left[s^2 + 2\delta_1 \frac{\omega_1}{\omega_0} s + \left(\frac{\omega_1}{\omega_0}\right)^2\right]\left[s^2 + 2\delta_2 \frac{\omega_2}{\omega_0} s + \left(\frac{\omega_2}{\omega_0}\right)^2\right] = 0 \tag{9}$$

The δ and ω parameters as a function of Vertistat inertia, spring, and damping parameters are as follows:

$$\frac{K_{D_1}''}{J_b''\omega_0} I_1 = 2\left(\delta_1 \frac{\omega_1}{\omega_0} + \delta_2 \frac{\omega_2}{\omega_0}\right) \tag{10a}$$

$$\frac{K_{s_1}''}{J_b''\omega_0^2} I_1 - 3\frac{J_a'}{J_b'} = 4\delta_1\delta_2 \frac{\omega_1}{\omega_0}\frac{\omega_2}{\omega_0} + \left(\frac{\omega_1}{\omega_0}\right)^2 + \left(\frac{\omega_2}{\omega_0}\right)^2 \tag{10b}$$

$$3\frac{K_{D_1}''}{J_b''\omega_0} I_2 = 2\frac{\omega_1}{\omega_0}\frac{\omega_2}{\omega_0}\left(\delta_1 \frac{\omega_2}{\omega_0} + \delta_2 \frac{\omega_1}{\omega_0}\right) \tag{10c}$$

$$3\frac{K_{s_1}''}{J_b''\omega_0^2} I_2 - 9\left(1 - \frac{J_a'}{J_b'}\right) = \left(\frac{\omega_1\omega_2}{\omega_0^2}\right)^2 \tag{10d}$$

where $I_2 \equiv (1 - J_b''/J_b' - J_a'/J_b')$ and $I_1 \equiv 1 + J_b''/J_b'$ are used to simplify the presentation.

The stability analysis amounts to evaluation of δ and ω as functions of Vertistat parameters. When the assumption is made that $\omega_1 = \omega_2$ and $\delta_1 = \delta_2$, the equations simplify tremendously. Results of this particular case only are presented in this paper. This assumption is not made merely to obtain mathematical simplicity. Accumulated analog computer transient data directed toward a search for optimum combination of parameters has shown that this case is at or near optimum.

When $\delta_1 = \delta_2 = \delta$, equations fixing the damping and spring parameters in terms of rod and satellite inertias and damping ratio are derived from Eq. (10) to be

$$\frac{K_{s_1}''}{J_b''\omega_0^2} = \frac{3}{I_2}\left[1 - \frac{J_a'}{J_b'} + \left(\frac{1 - (J_a'/J_b') - (J_a''/J_b'')}{I_1}\right)^2\right] \tag{11}$$

$$\frac{K_{D_1}''}{J_b''\omega_0} = \frac{2\delta(3)^{1/2}}{I_1}\left[-\frac{J_a'}{J_b'} + \frac{I_1K_{s_1}''}{3J_b''\omega_0^2} - \frac{2I_2(2\delta^2 - 1)}{I_1}\right] \tag{12}$$

When in addition $\omega_1 = \omega_2 = \omega$, then

$$\delta = \frac{1}{2I_2}\left(\frac{J_b''}{J_b'}\right)^{1/2}\left(2 - \frac{J_a'}{J_b'}\right) \tag{13}$$

$$\omega/\omega_0 = (3I_2/I_1)^{1/2} \tag{14}$$

The satellite to primary rod inertia ratio, J_a'/J_b', will be small as compared to unity and usually can be ignored. If this assumption is made, and when Eq. (13) is substituted in Eq. (12), the damping and spring constants are dependent upon α to primary rod inertia ratio only (J_b''/J_b'). This is the primary quantity in determining stability.

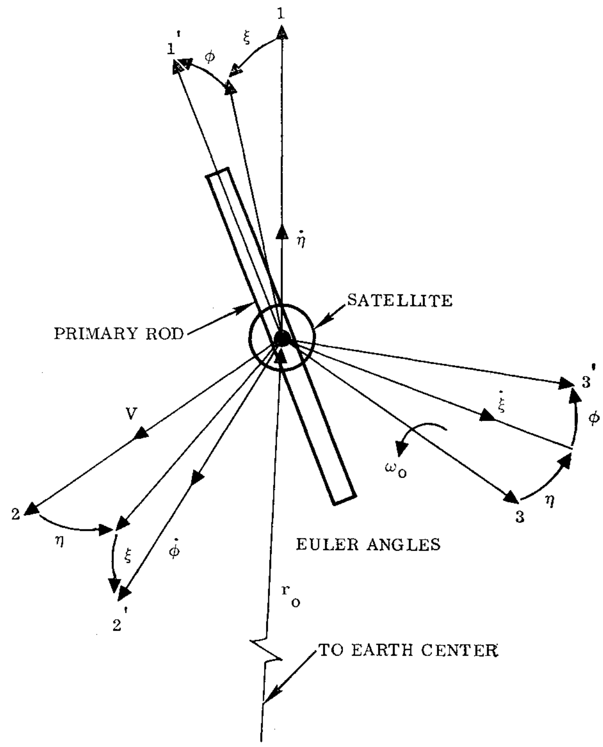

Fig. 2 Coordinate frames.

Routh's stability criteria, applied to the system characteristic Eq. (9), gives the following restraint on spring parameter magnitude:

$$K_{s_1}''/J_b''\omega_0^2 > (3/I_2)[1 - (J_a'/J_b')] \tag{15}$$

This result is immediately obtained from Eq. (10d), because $(\omega_1\,\omega_2)^2$ cannot be less than zero. This was the only condition required to attain stability. For all other combinations of parameters the stability could be poor but would not be divergent.

Figure 4 shows plots of the selected values of spring and damper parameters [Eqs. (11) and (12)] and the minimum

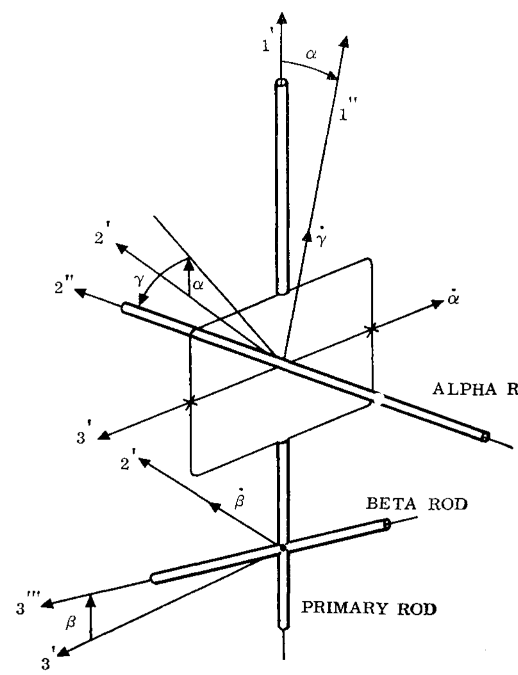

Fig. 3 Six-degree-of-freedom Vertistat nomenclature.

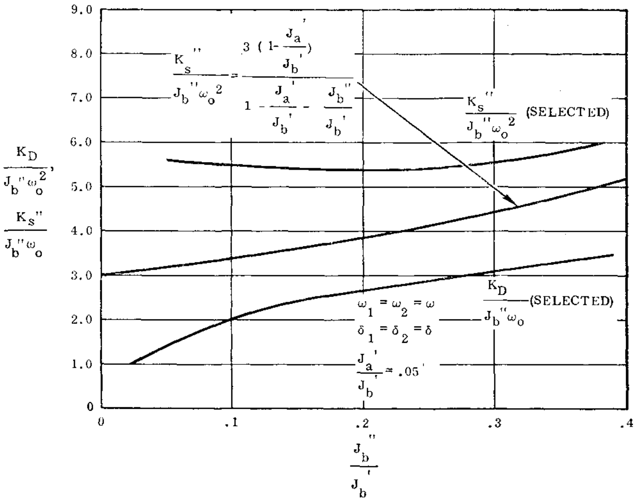

Fig. 4 Spring and damping constants vs rod inertia ratio.

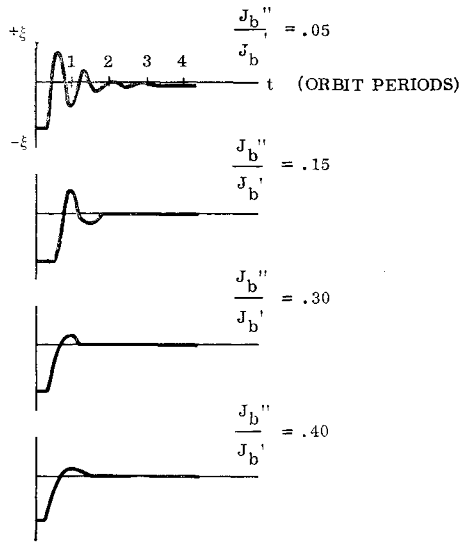

Fig. 6 Pitch analog data.

permissable value of spring parameter from Eq. (15). Figure 5 is a plot giving the damping ratio δ and the time factor $T = \omega_0/2\pi\delta\omega$ as a function of J_b''/J_b'. The time factor is the time in orbit periods required for the damped sinusoid to reduce to 37% of its initial value, a highly significant design parameter. From Fig. 5 it is seen that both damping ratio and the time factor change in a desirable direction as the secondary to primary boom ratio is increased. A series of pitch response time histories for different values of J_b''/J_b' is given in Fig. 6. As expected, the trend in Fig. 5 and Fig. 6 data, in regard to reduction in damping time as the ratio J_b''/J_b' increases, is identical.

Roll-Yaw Stability

The calculation of stability parameters for roll yaw is rigorously determined by simultaneous solution of Eqs. (5–8). However, certain approximations allow rapid determination of approximate results. Consider Eqs. (5) and (6) only, ignoring all terms not containing either the β or ϕ variables. This assumption removes a slight cross-coupling of yaw and roll. Compare these equations to the pitch or planar set of equations and it will be seen that the roll and pitch equations

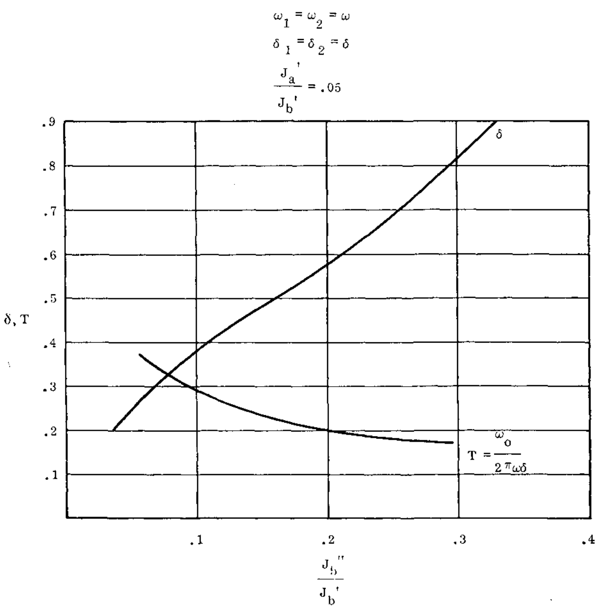

Fig. 5 Damping time factor τ and damping ratio δ vs rod inertia ratio.

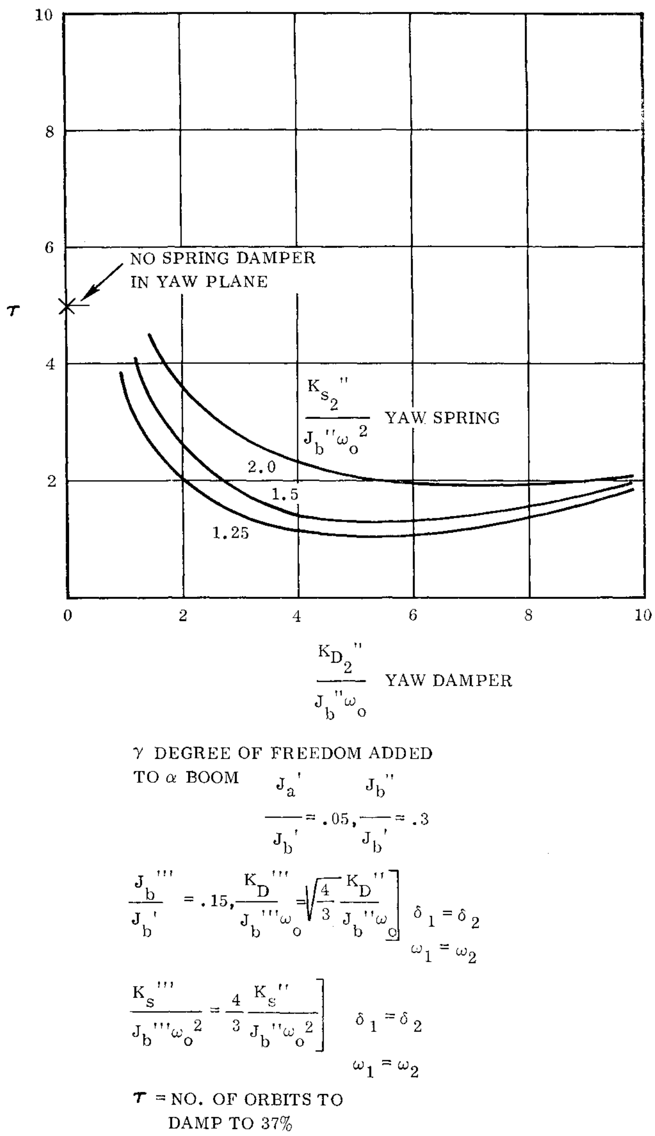

Fig. 7 Yaw error decay time vs damping and spring constants.

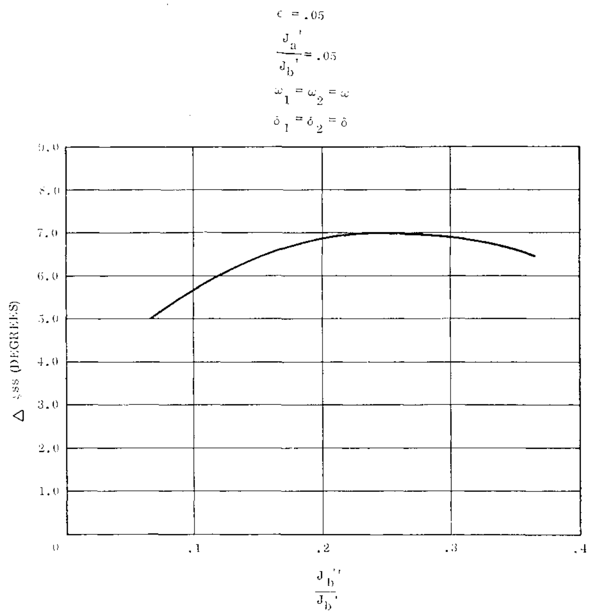

Fig. 8 Vertical error due to eccentricity vs rod inertia ratio.

are identical except for the "4" and "3" coefficients. If we make a further time base transformation

$$\tau\phi = (\tfrac{4}{3})^{1/2}\tau \quad \text{or} \quad \ddot{\phi}_\phi = \tfrac{3}{4}\ddot{\phi}, \dot{\phi}_\phi = (\tfrac{3}{4})^{1/2}\dot{\phi} \tag{16}$$

then Eqs. (5) and (6) are identical in form to those of the planar case. When the damping and spring constants for roll are set to

$$\begin{aligned} K_s'''/J_b''' &= \tfrac{4}{3}(K_{s1}''/J_b'') \\ K_D'''/J_b''' &= (\tfrac{4}{3})^{1/2}(K_{D1}''/J_b'') \end{aligned} \tag{17}$$

the results for roll match that for pitch. This means that the results given for pitch are applicable to roll if the time base in pitch is changed according to Eq. (16). The slight change in time base could be considered trivial and the rough conclusion drawn that roll and pitch performance is essentially identical.

There is no apparent way to quickly determine yaw stability from inspection of equations. The procedure adopted was to first simulate (analog computer) Eqs. (5–7). Elimination of (8) and deletion of the γ terms from (7) amounts to removal of the γ degree of freedom of the α boom relative to the primary boom (no spring damper in yaw plane). Then the spring damper in yaw was added to determine the value of this additional hardware and suitable parameter settings for same.

Typical results are shown in Fig. 7. All three boom inertias and roll-yaw spring-damper constants were set at a configuration determined by past analysis[1] to be acceptable from a roll-yaw stability viewpoint. For these settings, the time factor in yaw was five orbit periods without the additional spring damper in yaw. When they were added the yaw time factor improved as indicated. By suitable selection of yaw spring-damper parameters, the time factor could be improved by a factor of five.

Orbit Eccentricity

For the small disturbance case under discussion, orbit eccentricity introduces a driving torque to cause a pitch error only. The torque is due to the inertial reaction of the Vertistat primary and α rods to the sinusoidal variation with time of the angular rate of local vertical. Quantitatively, this torque T_0 is

$$\begin{aligned} T\epsilon' &= 2J_b'\epsilon \sin\omega_0 t \text{ (primary rod)} \\ T\epsilon'' &= 2J_b''\epsilon \sin\omega_0 t \text{ (α rod)} \end{aligned} \tag{18}$$

where ϵ is orbit eccentricity. In order to evaluate the magnitude of pitch error resulting from these torques, Eqs. (3) and (4) are solved for a steady-state solution (ignoring transient terms). The maximum magnitude of the steady-state sinusoidal variation (at orbit period) of the pitch angle is

$$\Delta\xi_{ss}\,(\text{deg}) = \frac{2\epsilon I_1\, 57.3}{2 - \dfrac{4J_b''}{J_b'} - 3\dfrac{J_a'}{J_b'}} \times \left[\frac{[(K_{s1}''/J_b''\omega_0^2) - (4/I_1)]^2 + (K_{D1}''/J_b''\omega_0)^2}{\left(\dfrac{K_{D1}''}{J_b''\omega_0}\right)^2 + 4\left(\dfrac{4[2 - 3\,(J_b''/J_b')]}{2 - 4(J_b''/J_b') - 3(J_a'/J_b')} - \dfrac{K_{s1}''}{J_b''\omega_0^2}\right)^2}\right] \tag{19}$$

With the spring and damper parameters set to yield constant stability independent of altitude (this means that $K_{s1}''/J_b''\omega_0^2$ and $K_{D1}''/J_b''\omega_0$ are constant for all altitudes), the steady-state pitch error is then linearly dependent upon eccentricity but independent of altitude. Figure 8 plots $\Delta\xi_{ss}$ as a function of α to primary boom inertia ratio for orbit eccentricity equal to 0.05. As indicated, there is little significant change as a function of boom parameters. Roughly 1.5° of steady-state error per 0.01 orbit eccentricity occurs.

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